Jost-Lehmann-Dyson Representation, Analyticity in Angle Variable and Upper Bounds in Noncommutative Quantum Field Theory

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Abstract

The existence of Jost-Lehmann-Dyson representation analogue has been proved in framework of space-space noncommutative quantum field theory. On the basis of this representation it has been found that some class of elastic amplitudes admits an analytical continuation into complex $\cos\vartheta$ plane and corresponding domain of analyticity is Martin ellipse. This analyticity combined with unitarity leads to Froissart-Martin upper bound on total cross section.

1 Introduction

The proof of analytical properties of elastic scattering amplitudes in $\cos\vartheta$, (ϑ is a scattering angle) is one of the most important achievements of quantum field theory (QFT). The first step was done by Lehmann [1], who proved that $\pi - N$ elastic scattering amplitude is an analytical function of $\cos\vartheta$ in some ellipse (Lehmann ellipse). Martin [2] derived that using analyticity with respect to energy variable it is possible to enlarge sufficiently the above mentioned domain of analyticity. The exact size of this domain was established in [3, 4]. This domain is named Martin ellipse.

The analyticity in $\cos\vartheta$ plane together with unitarity leads to the very important bounds on high energy behaviour of scattering amplitude. The first rigorous bound, which follows from analyticity in Lehmann ellipse was obtained in the work of Greenberg and Low [5].

In accordance with this bound at $s \to \infty$

$$\sigma_{tot}(s) \leq C\, s\, \ln^2 \frac{s}{s_0}, \tag{1}$$

$\sigma_{tot}(s)$ is a total cross-section, C and s_0 are some constants. s is the usual variable, in this paper we deal only with a center of mass system.

Froissart [6] showed that the bound considerably stronger then (1) follows from double dispersion relations, namely

$$\sigma_{tot}(s) \leq C\, \ln^2 \frac{s}{s_0}. \tag{2}$$

Martin [7] first demonstrated that this bound is valid under much weaker condition on the domain of analyticity in $\cos\vartheta$ plane and then proved that the necessary domain of analyticity really exists in the framework of axiomatic QFT [2]. The bound (2) is named Froissart-Martin bound. The final step in derivation of the best axiomatic upper bound on total cross-section was done in [8], where it was proved that

$$\sigma_{tot}(s) \leq \frac{\pi}{m_\pi^2}\, \ln^2 \frac{s}{s_0}, \tag{3}$$

m_π is π-meson mass.

Precisely, slightly stronger bound can be obtained. Namely, at $s \to \infty$ (see, e.g. reviews [9] and [10])

$$\sigma_{tot}(s) \leq \frac{\pi}{m_\pi^2}\, \ln^2 \frac{s}{(\ln\, s)^{\frac{3}{2}}}. \tag{4}$$

The rigorous upper bounds were also found for a forward differential cross-section. The best bound of this kind was obtained by Singh and Roy [11]:

$$|F(s)| \leq \frac{s}{8\, m_\pi \sqrt{\pi}} \sqrt{\sigma_{tot}(s)}\, \ln \frac{s}{\sigma_{tot}(s)}. \tag{5}$$

The bounds for non-forward scattering were also obtained (see reviews [9, 12]).

At present noncommutative quantum field theory (NC QFT) is regarded as one of the most attractive possibilities to consider interaction at very short distances and so at very high energies. The study of such theories acquired an additional interest after it was shown that

they appear naturally, in some cases, as low-energy effective limits of open string theory [13]. In noncommutative quantum field theory the coordinate operators satisfy the commutation relations

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}, \tag{6}$$

where $\theta_{\mu\nu}$ is a constant antisymmetric matrix of dimension (length)^2.

The implications of the modern ideas of noncommutative geometry [14] in physics have been lately of great interest, though attempts can be traced back as far as 1947 [15]. Plausible new arguments for studying NC QFT have been offered in [13, 16] (for a review, see [17]). Thus it is very important to investigate analytical properties of scattering amplitudes in NC QFT.

We shall consider throughout this paper only the case of space-space noncommutativity, i.e. $\theta_{0i} = 0$, since theories with space-time noncommutativity can be obtained as low-energy effective limits from string theory only in special cases [18]. Besides, there are problems with unitarity [19] and causality [20, 21] in the general case.

In our papers [22, 23] it was shown that in space-space noncommutative field theory a forward elastic scattering amplitude has the same analytical properties as in commutative one. Let us point out that the first step in the derivation of similar analytical properties has been done in [24].

In commutative QFT the proof of analytical properties of elastic scattering amplitudes is based on the local commutativity (microcausality) condition, that is

$$[j_1(x), j_2(y)] = 0, \quad \text{if} \quad (x-y)^2 < 0, \tag{7}$$

where $j_i(x),\ i = 1, 2$, is an interacting current (see [25], eq. (3-34)). Below we take $j_1(x)$ to mean a nucleon current and $j_2(x)$ - a π-meson one.

In space-space NC field theory, that is $\theta_{0i} = 0$, we can, without lost of generality, consider the case, when only $\theta_{12} = -\theta_{21} \neq 0$ [26], in other words x_0 and x_3 are commutative variables and one can assume [26] that condition (7) has the noncommutative analogue:

$$[j_1(\hat{x}), j_2(\hat{y})] = 0, \quad \text{if} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \tag{8}$$

In our papers [22, 23] it was proved that condition (8) is sufficient to derive the usual forward dispersion relations. This fact would play a crucial role in obtaining the final domain of analyticity in angle plane in noncommutative case. We show that this domain is the same with the largest rigorously proved domain of such an analyticity in commutative case - Martin ellipse. This fact gives us the possibility to prove Froissart-Martin bound in case of space-space NC QFT. As we consider only this variant of noncommutative theory we do not mention below this point. The derivation of Jost-Lehmann-Dyson representation in NC space-space QFT was done also in the paper [27], which appeared simultaneously with the first version of this work. In [27] Froissart-Martin bound has been obtained, the result is based on the assumptions different from ours.

Our paper is organized as follows. We consider elastic scattering amplitude of two spinless particles with masses m (meson) and M (nucleon). We believe that, as well as in commutative case, obtained results would coincide with the results for $\pi - N$-scattering after averaging over spin. Our results can be also extended on other processes.

First we obtain the analogue of Jost-Lehmann-Dyson representation. Using this representation we derive the analyticity of the amplitude in question in Lehmann ellipse. Following the way proposed in [2] - [4] we can extend this ellipse up to Martin ellipse. In this extension as well as in commutative case is the existence of dispersion relations plays a crucial role.

In Appendix for the reader's convenience we give derivation of Froissart-Martin bound in its strongest form under the weakest assumptions.

Let us point out that in noncommutative case Froissart-Martin bound can be stronger than in the usual case (see (4)). The matter is that really Froissart-Martin bound contains the factor $\frac{\sigma_{el}(s)}{\sigma_{tot}(s)}$, where $\sigma_{el}(s)$ is an elastic scattering cross-section. In noncommutative case Froissart-Martin bound contains the analogous factor $\frac{\tilde{\sigma}_{el}(s)}{\sigma_{tot}(s)}$, where $\tilde{\sigma}_{el}(s)$ is an elastic cross-section in the case when momentums of initial particles are orthogonal to noncommutative plane.

2 Jost-Lehmann-Dyson Representation

Let us consider the matrix element

$$f(x) = \langle p' \,|\, \left[j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right)\right] |\, p \rangle, \tag{9}$$

where $|\,p\rangle$ and $|\,p'\rangle$ are arbitrary states with momentum p and p' correspondingly. To simplify notations we omit $\hat{}$ above x. Owing to condition (8)

$$f(x) = 0, \quad \text{if} \quad {x_0}^2 - {x_3}^2 < 0. \tag{10}$$

Fourier transformation of this matrix element is:

$$f(q) = \int e^{i\,q\,x}\, f(x)\, d\,x. \tag{11}$$

We omit unessential numerical factor. Carry out the integration in this expression over noncommutative variables x_1 and x_2 we obtain

$$f(q) \equiv f(q_0, q_3) = \int e^{i\,(q_0\,x_0 - q_3\,x_3)}\, f(x_0, x_3)\, d\,x_0\, d\,x_3. \tag{12}$$

Here

$$f(x_0, x_3) \equiv \int f(x)\, e^{-i\,(q_1\,x_1 - q_2\,x_2)}\, d\,x_1\, d\,x_2.$$

We do not write down the dependence $f(x_0, x_3)$ on q_1 and q_2. Here and in what follows (except as otherwise noted) q is a two dimensional vector $q = (q_0, q_3)$, $q^2 = {q_0}^2 - {q_3}^2$. The corresponding Fourier transformation is:

$$f(q) \equiv \int e^{i\,q\,x}\, f(x)\, d\,x_0\, d\,x_3, \quad f(x) \equiv f(x_0, x_3). \tag{13}$$

To use efficiently the condition (10) let us (similar to the commutative case) associate function $f(x)$ with the function $F(X)$ in four-dimensional space, where

$$F(X) = 4\,\pi\, f(x)\, \delta(X^2), \tag{14}$$

$$X_0 = x_0,\ X_1 = x_3,\ X_2 = y_1,\ X_3 = y_2, \quad X^2 = {x_0}^2 - {x_3}^2 - {y_1}^2 - {y_2}^2.$$

In accordance with condition (10)

$$f\,(x) = \frac{1}{4\,\pi^2} \int F\,(X)\, d^2\, y. \tag{15}$$

Indeed, by definition (14)

$$f\,(x) = \frac{1}{4\,\pi} \int F\,(X)\, d\, y^2 = \begin{cases} f\,(x) & \text{at} \quad x^2 \geq 0, \\ 0 & \text{at} \quad x^2 < 0 \end{cases} . \tag{16}$$

As in our case $f\,(x) = 0$ if $x^2 < 0$, we checked eq. (15).

Let us consider four-dimensional momentum space Q_i: $Q_0 = q_0,\ Q_1 = q_3,\ Q_2 = \lambda_1,$ $Q_3 = \lambda_2$; $Q^2 = {q_0}^2 - {q_3}^2 - {\lambda_1}^2 - {\lambda_2}^2$. It is easy to see that

$$\Box_4\, F\,(Q) = 0, \quad \Box_4 \equiv \frac{\partial^2}{\partial {q_0}^2} - \frac{\partial^2}{\partial {q_3}^2} - \frac{\partial^2}{\partial {\lambda_1}^2} - \frac{\partial^2}{\partial {\lambda_2}^2}, \tag{17}$$

where

$$F\,(Q) = \int e^{i\,Q\,X}\, F\,(X)\, d^4\, X.$$

The correspondence between $f\,(q)$ and $F\,(Q)$ can be easily obtained:

$$f\,(q) = F\,(\tilde{q}), \quad \text{where} \quad \tilde{q} = (q_0, q_3, 0, 0).$$

The following consideration is similar to the one in the usual (commutative) case with the only difference that now we have four-dimensional space instead of six-dimensional one. The general consideration in space of arbitrary dimensions was done in Vladimirov's book [28], see similar consideration in four-dimensional case in [29, 30].

In accordance with eq. (17) we can represent $F\,(Q)$ in the form:

$$F\,(Q) = \int\limits_{\Sigma} d\,\Sigma' \left[D\,(Q - Q')\, \frac{\partial F\,(Q')}{\partial \Gamma'} + F\,(Q')\, \frac{\partial D\,(Q - Q')}{\partial \Gamma'} \right], \tag{18}$$

where Σ is some three-dimensional hypersurface, $\frac{\partial}{\partial \Gamma'}$ is a conormal derivation on it and function $D\,(Q)$ satisfies eq. (17) as well as the following condition:

1. $$D\,(Q) \to 0 \quad \text{if} \quad Q_0 \to 0;$$

2. $$\frac{\partial D\,(Q)}{\partial Q_0} \to \delta\,(\vec{Q}) \quad \text{if} \quad Q_0 \to 0; \quad \delta\,(\vec{Q}) = \delta\,(Q_1)\,\delta\,(Q_2)\,\delta\,(Q_3);$$

3. $$\frac{\partial^2 D\,(Q)}{\partial {Q_0}^2} \to 0 \quad \text{if} \quad Q_0 \to 0.$$

The necessary function would be

$$D\,(Q) = \frac{-i}{(2\,\pi)^3}\int \epsilon\,(x_0)\,e^{i\,Q\,X}\,\delta\,(X^2)\,d^4\,X. \tag{19}$$

(The proof one can find in [28]).

We can, as well as in commutative case (see [29], Ch. 10), simplify eq. (18) integrating the second term by parts. Thus we can rewrite expression (18) as follows:

$$F\,(Q) = \int d^4\,Q'\,D\,(Q - Q')\,\psi\,(Q'). \tag{20}$$

At the moment it is convenient to consider integral in (19) formally as integral over all space (really $\psi\,(Q')$ contains the term $\delta\,(\Sigma)$). Below we see that integration limits are really determined by the properties of $f\,(q)$, which follow from translation invariance and spectral condition. Using the explicit expression for $D\,(Q)$ (see [28]):

$$D\,(Q) = \frac{1}{2\,\pi}\,\epsilon\,(Q_0)\,\delta\,(Q^2) \tag{21}$$

we come to the final expression:

$$F\,(Q) = \int d^4\,Q'\,\epsilon\,(Q_0 - Q'_0)\,\delta\,(Q - Q')^2\,\psi\,(Q'). \tag{22}$$

From eq. (22) it follows directly that $f\,(q) = F\,(\tilde{q})$ satisfies the representation:

$$f\,(q) = \int d^4\,u'\,\epsilon\,(q_0 - u'_0)\,\delta\,(q - u')^2\,\varphi\,(u') \tag{23}$$

(we change the notations: $Q' = u',\ \psi\,(Q') = \varphi\,(u')$).

Let us proceed to the similar expression for

$$f^r\,(q) = \int d\,q_0\,d\,q_3\,\tau\,(x_0)\,e^{i\,(q_0\,x_0 - q_3\,x_3)}\,f\,(x), \tag{24}$$

where

$$\tau\,(x_0) = 1, \quad x_0 \geq 0; \qquad \tau\,(x_0) = 0, \quad x_0 < 0.$$

It is easy to show that

$$f^r\,(q) = \frac{i}{2\,\pi}\int d\,q'_0\,\frac{f\,(q'_0, \vec{q})}{q'_0 - q_0}, \quad Im\,q_0 > 0. \tag{25}$$

Using expressions (24) and (23) we can easily make necessary integrations in (25) and finally obtain:

$$f^r\,(q) = \frac{i}{2\,\pi}\int d^4\,u'\,\frac{\varphi\,(u')}{(q - u')^2}, \quad Im\,q_0 > 0. \tag{26}$$

We represent space vector $\vec{u'}$ as the linear combination of two orthogonal vectors: $\vec{u}$, belonging to the plane, which contains the axis λ_1 and $\vec{q}$, and $\lambda_2\,\vec{e}$, where $\vec{e}$ is a unit vector, directed along the axis λ_2, $u'_0 \equiv u_0$. Eq. (26) can be written as follows

$$f^r\,(q) = \frac{i}{2\,\pi}\int d^3\,u\int d\,\lambda_2{}^2\,\frac{\varphi\,(u, {\lambda_2}^2)}{(q - u)^2 - {\lambda_2}^2}, \quad Im\,q_0 > 0. \tag{27}$$

Now we find the restrictions on the domain of integration in eqs. (23) and (27). To this end we use the spectral properties of $f\,(q)$. First let us represent $f\,(q)$ in eq. (11) as $f\,(q) = f_1\,(q) - f_2\,(q)$, where

$$f_1\,(q) = \int e^{i\,q\,x}\,\langle p'\,|\,j_1\left(\frac{x}{2}\right)\,j_2\left(-\frac{x}{2}\right)\,|\,p\rangle\,d\,x, \tag{28}$$

$$f_2\,(q) = \int e^{i\,q\,x}\,\langle p'\,|\,j_2\left(-\frac{x}{2}\right)\,j_1\left(\frac{x}{2}\right)\,|\,p\rangle\,d\,x. \tag{29}$$

Surely here q is a four dimensional vector.

Taking into account that translation invariance survives in NC QFT and using it as well as completeness of basic vectors $|P_n\rangle$, (P_n is the momentum of the n state) we obtain in Breit system, that is $\vec{p} + \vec{p'} = 0$:

$$f_1\,(q) = \sum_n \langle p'\,|\,j_1\,(0)\,|\,P_n\,\rangle\,\langle\,P_n\,|\,j_2\,(0)\,|\,p\rangle\,\delta\,(q_0 + a - \sqrt{M_n^2 + \vec{q}^2}) \tag{30}$$

$$f_2\,(q) = \sum_n \langle p'\,|\,j_2\,(0)\,|\,P_n\,\rangle\,\langle\,P_n\,|\,j_1\,(0)\,|\,p\rangle\,\delta\,(-q_0 + a - \sqrt{M_n^2 + \vec{q}^2}), \tag{31}$$

where $a = \frac{p_0 + p_0'}{2}$, M_n are masses of intermediate states.

Thus $f\,(q) = 0$ if the following double inequality is satisfied:

$$a - \sqrt{\vec{q}^2 + m_2^2} < q_0 < \sqrt{\vec{q}^2 + m_1^2} - a, \tag{32}$$

where m_1 and m_2 are the minimal masses of intermediate states $|\,P_n\,\rangle$. Just the same conditions are valid in commutative case [29].

Let us consider the case when $q_1 = q_2 = 0$. In this case we can use the same notation both for four-dimensional vector $(q_0, q_3, 0, 0)$ and two-dimensional one (q_0, q_3).

Condition (32) can be written in the form:

$$S_-\,(\vec{q}) < q_0 < S_+\,(\vec{q}), \quad \vec{q} = q_3. \tag{33}$$

We determine two surfaces $\sigma_\pm$, such that $q_0 = S_\pm\,(\vec{q})$.

In order to the condition (33) be satisfied automatically [29, 30] we choose as usual the domain of integration in (23) so that δ-function in (23) be zero, when q_0 satisfies the double inequality (33).

As it follows from eq. (23) $f\,(q) \neq 0$ only if

$$(q - u)^2 = {\lambda_2}^2. \tag{34}$$

Eq. (34) determines two-branch hyperboloid. Following Dyson let us call this hyperboloid admissible if its upper branch has no points below σ_+ and its lower one has no points upper σ_-, that is:

$$u_0 + \sqrt{(\vec{q} - \vec{u})^2 + {\lambda_2}^2} \geq S_+\,(\vec{q}),$$

$$u_0 - \sqrt{(\vec{q} - \vec{u})^2 + {\lambda_2}^2} \leq S_-\,(\vec{q}).$$

These conditions would be satisfied for any $\vec{q}$ if

$$u_0 \geq \max_q \left(S_+\left(\vec{q}\right) - \sqrt{\left(\vec{q}-\vec{u}\right)^2 + {\lambda_2}^2}\right),$$

$$u_0 \leq \min_q \left(S_-\left(\vec{q}\right) + \sqrt{\left(\vec{q}-\vec{u}\right)^2 + {\lambda_2}^2}\right).$$

The necessary calculations are similar to ones drawn in [29, 30]. As a result we have:

$$|u_0| + |\vec{u}| \leq a, \tag{35}$$

$${\lambda_2}^2 \geq max\{\left(m_2 - \sqrt{\left(a-u_0\right)^2 - \vec{u}^2}\right)^2, \left(m_1 - \sqrt{\left(a+u_0\right)^2 - \vec{u}^2}\right)^2\}. \tag{36}$$

3 Elastic Scattering Amplitude and Analyticity in Lehmann Ellipse

Let us consider $\pi - N$ elastic scattering amplitude for the process, in which π-meson with mass m has the initial momentum k and final k' and nucleon with mass M has the initial momentum p and final p'. As it was shown in [22] the usual Lehmann-Symanzik-Zimmermann (LSZ) formulas are valid in noncommutative space-space field theory. Thus up to the numerical factors and the term, which is a polynomial in energy, we can write

$$F\left(p', k', p, k\right) = \int e^{\frac{i\,(p-k)\,x}{2}} f^r\left(x\right) d\,x, \tag{37}$$

where

$$f^r\left(x\right) = \tau\left(x_0\right) \langle p', k' \mid \left[j_1\left(\frac{x}{2}\right), j_2\left(-\frac{x}{2}\right)\right] \mid 0\rangle,$$

$j_1\left(x\right)$ is a nucleon current and $j_2\left(x\right)$ is a π-meson current.

Let us consider scattering amplitude in the center of mass system. To use Jost-Lehmann-Dyson representation we represent scattering amplitude in the form:

$$F\left(p', k', p, k\right) = \int e^{i\left[\frac{p_0-k_0}{2}\,x_0 - p_3\,x_3\right]} f^r\left(x_0, x_3\right) d\,x_0\,d\,x_3, \tag{38}$$

where

$$f^r\left(x_0, x_3\right) = \int e^{-i\,(p_1\,x_1 + p_2\,x_2)} f^r\left(x\right) d\,x_1\,d\,x_2.$$

As before we consider the case $p_1 = p_2 = 0$.

To derive analyticity in Lehmann ellipse let us use the representation (27) for elastic scattering amplitude

$$F\left(p', k', p, k\right) = \frac{i}{2\,\pi} \int d^3\,u \int\limits_{\min\,{\lambda_2}^2}^{\infty} d\,{\lambda_2}^2\, \frac{\varphi\left(u, {\lambda_2}^2, p', k', \theta_{\mu\nu}\right)}{\left(u_0 - \frac{p_0-k_0}{2}\right)^2 - \left(\vec{u}-\vec{p}\right)^2 - {\lambda_2}^2}. \tag{39}$$

We choose axis λ_1 so that it would belong to the plane formed by vectors $\vec{q}$ and $\vec{p'}$. Let us denote the angle between $\vec{u}$ and $\vec{p'}$ as α, then $\vec{u}\,\vec{p} = |\vec{u}|\,|\vec{p}|\,\cos(\vartheta - \alpha)$, where ϑ is the angle between $\vec{p}$ and $\vec{p'}$. Eq. (39) is similar to commutative one [12] with the only and evident difference that in our case we have no additional integrations. Let us stress that in noncommutative case numerator has an additional dependence of $\theta_{\mu\nu}$, but similarly to the usual case all dependence of $\cos\vartheta$ is contained in denominator.

Now let us consider the dominator in the last integral in (39). The direct calculation shows that denominator is

$$-2\,|\vec{u}|\,|\vec{p}|\,\sin\,\beta\,[y - \cos(\vartheta - \alpha)],$$

where

$$y = \frac{\vec{u}^2 + \vec{p}^2 + {\lambda_2}^2 - \left(u_0 - \frac{p_0 - k_0}{2}\right)^2}{2\,|\vec{u}|\,|\vec{p}|\,\sin\,\beta}.$$

Now our goal is to calculate the minimal value of $y \equiv y_0(s)$. It is significant that calculations in noncommutative case reduce to the similar one in commutative case when $p_1 = p_2 = 0$. In other words, we consider elastic scattering process in which $\vec{p}$ is orthogonal to noncommutative plane. As forward scattering amplitude is a function of s only, it does not depend on this additional condition.

As above we use translation invariance and spectral condition in order to obtain the necessary constrains on $y_0(s)$. Actually, repeating the steps we have made in derivation of double inequality (32) we obtain the similar one, but now $a = ({p_0}' + k_0)'/2$. Evidently the constrains (35) and (36) are valid. To estimate minimum of y we can use directly the results obtained in commutative case. Indeed, the restrictions (35), (36) contain only $\vec{u}^2$. This minimum is searched among all possible values of $\vec{u}$, which include the case when $\vec{u}$ belongs the plane containing vectors $\vec{p}$ and $\vec{p'}$, that actually is our case. As in commutative case the analogous minimum is realized just in case when this additional condition is satisfied [1], the results coincide both in commutative case and noncommutative one. The derivation of this minimum also uses the masses of the lowest intermediate states, which are evidently the same with commutative case. As this minimum is more than one the right-hand side of eq. (39) is well-defined also for nonphysical $\cos\vartheta$, resulting in analyticity of the amplitude under consideration with respect to $\cos\vartheta$ in the domain, which is Lehmann ellipse. Let us recall that Lehmann ellipse is the ellipse with focuses in points ± 1 and with major half-axis $y_0(s)$. The exact value of $y_0(s)$ one can find in [12], here we only mentioned that at $s \to \infty$

$$y_0(s) \cong 1 + \frac{c(m, M)}{s^2} \quad \text{i.e.} \quad t_0(s) \cong \frac{2\,c(m, M)}{s}.$$

Let us remind that at high energies transfer momentum is represented by scattering angle as follows: $\cos\vartheta = 1 + 2\,t/s$.

Using unitarity and spectral properties of scattering amplitude it is easy to show that as in commutative case imaginary part of elastic scattering amplitude - $A(s, \cos\,\vartheta)$ is an analytical function in the domain, which is larger than analogous domain for $F(s, \cos\,\vartheta)$ - large Lehmann ellipse [1], see also [12].

4 Unitary Constrains on Particle Amplitudes

For the elastic process, when $p_1 = p_2 = 0$, we can say that scattering amplitude is a function of s and $\cos\vartheta$ (the dependence of this function on $\theta_{\mu\nu}$ does not change the results derived below. Thus for the elastic scattering amplitude in question we can write

$$F\,(p', k', p, k) = F\,(s, \cos\,\vartheta, \theta_{\mu\nu}) \equiv F\,(s, \cos\,\vartheta).$$

We can expand $F\,(s, \cos\,\vartheta)$ in the Legendre polynomials:

$$F\,(s, \cos\,\vartheta) = 2\sum_{0}^{\infty}(2\,l+1)\,\tilde{f}_l\,(s) P_l\,(\cos\,\vartheta) \tag{40}$$

as $F\,(s, \cos\,\vartheta)$ is an analytical function in Lehmann ellipse [31].

As we further deal with $F\,(s, \cos\,\vartheta)$ at $s \to \infty$, we substitute the factor in front of the sum by its asymptotic value.

Surely $\tilde{f}_l\,(s)$ depends on $\theta_{\mu\nu}$, but this dependence does not change constrains, which we obtain below. We use the notation $\tilde{f}_l\,(s)$ instead of the usual $f_l\,(s)$ in order to emphasize that we consider only the specific class of elastic scattering amplitudes. Owing to orthogonality of the Legendre polynomials

$$\tilde{f}_l\,(s) = \int_{-1}^{1} F\,(s, \cos\,\vartheta)\, P_l\,(\cos\,\vartheta)\, d\,\vartheta. \tag{41}$$

Let us point out that for forward scattering eq. (40) is the general expression as $F\,(s, 1)$ does not depend on the direction of $\vec{q}$. Now let us consider the special differential cross section and thus special elastic scattering with $\tilde{\sigma_{el}}\,(s)$, corresponding to the chosen class of amplitudes. That is

$$s\,\tilde{\sigma_{el}}\,(s) = \int_{-1}^{1} F\,(s, \cos\,\vartheta) F^*\,(s, \cos\,\vartheta)\, d\,\cos\,\vartheta. \tag{42}$$

Using the eq. (40), we obtain

$$s\,\tilde{\sigma_{el}}\,(s) = \sum_{0}^{\infty}(2\,l+1)\,|\tilde{f}_l\,(s)|^2 \tag{43}$$

In accordance with the optical theorem, which follows only from the unitarity and thus has to be also valid in NC case we have:

$$s\,\sigma_{tot}\,(s) = \sum_{0}^{\infty}(2\,l+1)\,\tilde{a}_l\,(s), \quad \tilde{a}_l\,(s) \equiv Im\,\tilde{f}_l\,(s), \tag{44}$$

we use here the convenient normalization for $\sigma_{tot}\,(s)$ and $\sigma_{el}\,(s)$.

As

$$\tilde{\sigma_{el}}\,(s) < \sigma_{tot}\,(s)$$

we obtain that

$$\tilde{a}_l\,(s) \geq |\tilde{f}_l\,(s)|^2, \tag{45}$$

which is the usual unitarity constrain on partial amplitudes. Thus we have no problem with the derivation of Froissart-Martin bound. Let us point out that Froissart-Martin bound in NC case can be stronger than in the usual one as we deal only with $\tilde{\sigma_{el}}\,(s)$ (see the end of Introduction).

5 Martin Ellipse and Froissart-Martin Bound in NC QFT

The possibility to enlarge Lehmann ellipse up to the Martin's one is caused on the one hand by unitary constrains on partial amplitudes and on the other hand by dispersion relations (DR). Let us remind that Martin ellipse has focuses at the same point as the Lehmann's one, its extremely right point is $t_0\,(s) = 4\,m_\pi^2$.

Strictly speaking the validity of DR is necessary not only for forward direction, but also for arbitrary small negative t, which is the usual variable. In noncommutative case DR were proved only for forward direction [22, 23]. But in accordance with proved analyticity of $F\,(s, \cos\,\vartheta)$ in Lehmann ellipse DR are valid also for some range of negative t.

The necessary unitary constrains on partial amplitudes were derived in previous section. From inequality (45) it follows directly that

$$0 \leq \tilde{a}_l\,(s) \leq 1. \tag{46}$$

In accordance with formula (40) we have:

$$A\,(s, \cos\,\vartheta) = 2\,\sum_0^\infty\,(2\,l+1)\,\tilde{a}_l\,(s)\,P_l\,(\cos\,\vartheta). \tag{47}$$

From condition (46) and the Legendre polynomial properties it follows directly that

$$\left.\frac{d^n\,A\,(s, \cos\,\vartheta)}{(d\,\cos\,\vartheta)^n}\right|_{\cos\,\vartheta=1} \geq 0. \tag{48}$$

The last condition plays a principal role in Martin method of enlarging Lehmann ellipse.

One point has to be mentioned. In the extension of Lehmann ellipse, Martin used the results of Bros, Epstein and Glaser [32]. Let us recall that in this paper analyticity in $\cos\,\vartheta$ were proved for nonphysical s. Here we derived analyticity only for physical values of s. Nevertheless following to Sommer [3], Bessis and Glaser [4] we can use for this purpose analyticity in Lehmann ellipses only. Thus we have no problem with the proof of analyticity of $F\,(s, \cos\,\vartheta)$ in Martin ellipse and so Froissart-Martin bound is valid in space-space NC QFT.

As it was first found by Martin [33] upper bound (2) in fact contains an additional factor $\sigma_{el}\,(s)/\sigma_{tot}\,(s)$. The strongest bound of such a kind was obtained in [11]. Namely,

$$\sigma_{tot}\,(s) \leq \frac{\pi}{m_\pi^2}\,\frac{\sigma_{el}\,(s)}{\sigma_{tot}\,(s)}\,\ln^2\,\frac{s}{(\ln\,s)^{\frac{3}{2}}}. \tag{49}$$

In noncommutative case we have also the additional factor, but now

$$\sigma_{tot}(s) \leq \frac{\pi}{m_\pi^2} \frac{\tilde{\sigma}_{el}(s)}{\sigma_{tot}(s)} \ln^2 \frac{s}{(\ln s)^{\frac{3}{2}}}. \tag{50}$$

Thus in general noncommutative Froissart-Martin bound may have an additional factor, which is less than unity.

In conclusion let us point out that inequality (5) can be also proved by standard way. Bounds at $t < 0$ can be obtained directly only for considered class of scattering amplitudes.

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6 Appendix: Derivation of Froissart-Martin Bound

We start with very weak upper bound on $A(s, t_0),\ t_0 = 4\,m^2 - \epsilon$:

$$A(s, t_0) < \exp(s^n), \qquad s \to \infty. \tag{A.1}$$

It was shown by Logunov, Nguyen van Hieu and Todorov [34] that this condition is a sufficient one for derivation of polynomial boundedness of elastic scattering amplitudes at physical energies.

Indeed, at $s \to \infty$ and $l/\sqrt{s} \to \infty$

$$P_l(x_0) \cong \frac{e^\gamma}{\sqrt{2\,\pi\,\gamma}}, \quad \gamma \equiv 2l\sqrt{\frac{t_0}{s}}, \quad x_0 = 1 + \frac{2t_0}{s} \tag{A.2}$$

(see [9, 10]).

Let us recall that in accordance with analyticity of $A(s, t)$ in Martin ellipse the series (47) converges at $t = t_0$. All terms in this series are positive according to condition (46) since $P_l(x) > 1$ if $x > 1$. Owing to (A.2) bound (A.1) can be satisfied only if

$$\tilde{a}_{l'+L} < \exp\left(-2\,l'\sqrt{\frac{t_0}{s}}\right), \tag{A.3}$$

where $L \sim s^{n+1/2}$.

Using constrains (46) on $\tilde{a}_l(s)$ for $l \leq L$ and inequality (A.3) for $l \geq L$, it is easy to estimate

$$A(s, 0) = 2\sum_0^\infty (2\,l + 1)\,\tilde{a}_l(s) \tag{A.4}$$

and obtain polynomial boundedness of $A(s, 0)$.

In accordance with Jin and Martin result [35] polynomial boundedness of $A(s, 0)$ leads to polynomial boundedness of $A(s, t_0)$ as number of subtractions in DR coincides at $t = 0$ and $t = t_0$ (if this number is even). Polynomial boundedness of $A(s, t_0)$ leads to the new constrains on $\tilde{a}_l(s)$.

Precisely, if $A\,(s, t_0) < s^k$ then inequality (A.3) is satisfied if

$$L = \frac{k - 1/2}{2} \sqrt{\frac{s}{t_0}} \,\ln\, s. \tag{A.5}$$

Repeating the above mentioned calculations we obtain Froissart-Martin bound, but with an unknown constant instead of π/m_π^2. Following this way and using now $k = 2$, we come to inequality (3) with additional factor 9/4.

To obtain the strongest upper bound first notice that condition

$$\int_{s_0}^{\infty} \frac{A\,(s, t_0)\,\delta\, s}{s^3} < \infty \tag{A.6}$$

implies that

$$A\,(s, t_0) < \frac{s^2}{\ln\, s}, \qquad s \to \infty. \tag{A.7}$$

Taking into account that max $A\,(s, 0)$ is a growing function of $A\,(s, t_0)$, we obtain the desired bound if we find the set of $\tilde{a}_l\,(s)$ that realize max $A\,(s, 0)$ at given value of $A\,(s, t_0)$. Let us show that this set is:

$$\tilde{a}_l\,(s) = \begin{cases} 1 & l \le L \\ \eta \le 1 & l = L + 1 \\ 0 & l \ge L + 2 \end{cases} \tag{A.8}$$

This result follows from the property of the Legendre polynomials: $P_{l_2}\,(x) > P_{l_1}\,(x)$, if $x > 1$ and $l_2 > l_1$.

Let us prove that any set of $\tilde{a}_l\,(s)$ different from the set (A.8) can not realize max $A\,(s, 0)$. Really in any other set there always exist two partial amplitudes $\tilde{a}_{l_1}\,(s)$ and $\tilde{a}_{l_2}\,(s)$, $l_2 > l_1$, such that $\tilde{a}_{l_1}\,(s) < 1$ and $\tilde{a}_{l_2}\,(s) > 0$. Let us replace $\tilde{a}_{l_1}\,(s)$ by $\tilde{a}_{l_1}\,(s) + \Delta_1$ and $\tilde{a}_{l_2}\,(s)$ by $\tilde{a}_{l_2}\,(s) - \Delta_2$, $\Delta_i > 0$ in such a way that $A\,(s, t_0)$ remains unchanged, i.e.

$$(2\,l_1 + 1)\,\Delta_1 - (2\,l_2 + 1)\,\Delta_2\,\frac{P_{l_2}\,(x_0)}{P_{l_1}\,(x_0)} = 0.$$

It is evident that

$$\Delta\, A\,(s, 0) = (2\,l_1 + 1)\,\Delta_1 - (2\,l_2 + 1)\,\Delta_2 > 0.$$

In order to find L we note that the contribution from the partial amplitude with $l = L + 1$ can be neglected, and because of the known recursion formula

$$(2\,l + 1)\,P_l\,(x) = P^{'}_{l+1}\,(x) - P^{'}_{l-1}\,(x)$$

the Legendre polynomial series of $A\,(s, t_0)$ can be summed up. As a result we have the following equation on L:

$$A\,(s, t) = P'_{L+1}\,(x) + P'_L\,(x) \cong 2\,P'_L\,(x) \tag{A.9}$$

From (A.2) it follows that

$$P'_l\left(1 + \frac{2t_0}{s}\right) \cong \frac{e^\gamma\,\sqrt{\gamma}}{4\,\sqrt{2\pi}}\,\frac{s}{t_0}, \qquad \gamma = 2L\sqrt{\frac{t_0}{s}}. \tag{A.10}$$

Thus in accordance with (A.9) and (A.7)

$$e^{\gamma}\sqrt{\gamma} \cong \frac{s}{\ln\, s}. \tag{A.11}$$

This equation is easily solved by the method of successive approximations. As a result we obtain that at $s \to \infty$

$$\gamma \cong \ln\, \frac{s}{(\ln\, s)^{\frac{3}{2}}}. \tag{A.12}$$

(For the details see [10]).

According to the optical theorem we have at $t = 0$:

$$\sigma_{tot}^{max}\,(s) \cong \frac{32\,\pi}{s} \sum_{l=0}^{L} 2\,l \cong \frac{16\,\pi}{s}\,L^2 = \frac{4\,\pi}{t_0}\,\gamma^2. \tag{A.13}$$

Taking into account (A.12) we see that equality (A.13) implies that desired inequality (4) is fulfilled.

Let us point out that maximum of $\sigma_{tot}\,(s)$ is reached if $\sigma_{el}\,(s) = \sigma_{tot}\,(s)$, see eqs. (43) - (45).

References

[1] H. Lehmann, *Nuovo Cim.* **10** (1958) 579.

[2] A. Martin, *Nuovo Cim.* **42** (1966) 930.

[3] G. Sommer, *Nuovo Cim.* **48A** (1967) 92; **52A** 373; 850; 866.

[4] J. D. Bessis and V. Glaser, *Nuovo Cim.* **50A** (1967) 568.

[5] O. W. Greenberg, F. E. Low, *Phys. Rev.* **124** (1961) 2047.

[6] M. Froissart, *Phys. Rev.* **123** (1961) 1053.

[7] A. Martin, *Phys. Rev.* **129** (1963) 1432.

[8] L. Łukashúk and A. Martin, *Nuovo Cim.* **52A** (1967) 122.

[9] S. M. Roy, *Physics Reports* **5C** (1972) 125.

[10] Yu. S. Vernov and M. N. Mnatsakanova, *Phys. of Particles and Nuclei*, **32** (2001) 589, Sect. 6.

[11] V. Singh and S. M. Roy, *Ann. of Phys.* **57** (1970) 461.

[12] A. A. Logunov, M. A. Mestvirishvili, O. A. Khrustalev, *Particles and Nuclei* **3** (1972) 1.

[13] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 32, hep-th/9908142.

[14] A. Connes, *Noncommutative Geometry*, Academic Press, New York, 1994.

[15] H. S. Snyder, *Phys. Rev* **71** (1947) 38.

[16] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Phys. Lett* **B331** (1994) 39; *Comm. Math. Phys.* **172** (1995) 187.

[17] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977, hep-th/0106048.

[18] O. Aharony, J. Gomis and T. Mehen, hep-th/0006236 (2000).

[19] J. Gomis and T. Mehen, *Nucl. Phys.* **B 591** (2000) 265, hep-th/0005129.

[20] N. Seiberg, L. Susskind and N. Toumbas, *JHEP* **0006** (2000) 044, hep-th/0005015; L. Álvarez-Gaumé and J. L. F. Barbon, *Int. J. Mod. Phys* **A 16** (2001) 1123, hep-th/0006209.

[21] M. Chaichian, K. Nishijima and A. Tureanu, *Phys. Lett.* **B 568** (2003) 146, hep-th/0209006.

[22] M. Chaichian, M. Mnatsakanova, A. Tureanu and Yu. Vernov, *Nucl. Phys.* **B673** (2003) 476, hep-th/0306158, 2003, 27.

[23] Yu. Vernov, M. Mnatsakanova, *Teor. Mat. Fiz.* **139** (2004) 3.

[24] Yi Liao, K. Sibold, *Phys. Lett. B.* **549** (2002) 352, hep-th/0209221.

[25] R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics and all that.* W. A. Benjamin, New York - Amsterdam, 1964.

[26] L. Álvarez-Gaumé and J. L. F. Barbon, *Int. J. Mod. Phys* **A 16** (2001) 1123, hep-th/0006209.

[27] M. Chaichian and A. Tureanu, hep-th/0403032, 2004, 28.

[28] V. S. Vladimirov, *Methods of the Theory of Functions of Several Complex Variables*, Cambridge, Massachusetts, MIT Press, 1966, Ch. 5.

[29] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley, New York, 1980, 3rd ed., Ch. 10.

[30] S. S. Schweber, *An introduction to relativistic quantum field theory*, Row, Peterson and Co, New York, 1961, Ch. 18.

[31] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, 1962, Sect 15.41.

[32] J. Bros, H. Epstein and V. Glaser, *Nuovo Cim.* **31** (1964) 1264;
Comm. Math. Phys. **1** (1965) 240.

[33] A. Martin, *Nuovo Cim.* **29** (1963) 993.

[34] A. A. Logunov, Nguyen Van Hieu and I. T. Todorov, *Usp. Fiz. Nauk* **88** (1966) 51 (in Russian).

[35] Y. S. Jin and A. Martin, *Phys. Rev.* **135B** (1964) 1375.